First midsemestral examination 2008
M.Math. II — Commutative algebra
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Throughout A stands for a commutative ring with $1 \neq 0$.
Be brief !

Q 1.
Show that every finitely presented, flat A-module is projective.

OR

If A is a domain in which each finitely generated ideal is principal, show that a module is flat if and only if it is torsion-free.

Q 2.
Let $0 \to L \to M \to N \to 0$ be a short exact sequence of A-modules where M is finitely generated and N is finitely presented. Prove that L must be finitely generated.

OR

Prove that a finitely generated, projective module over a local ring is necessarily free.

Q 3.
In $A = K[X, Y]$ where K is a field, show that the ideals $(X), (Y), (XY)$ which are all free A-modules. Show also that $(X) + (Y)$ is not a flat A-module.

Q 4.
(i) Let Q be a primary ideal in A. If $a \notin Q$, show that both Q and $(Q : a)$ are $\sqrt{Q}$-primary. What is $\sqrt{(Q : a)}$ in this case?
(ii) Let I be an ideal of A and let $I = \bigcap_{i=1}^{r} Q_i$ be a reduced primary decomposition. Show that, for each $i \leq r$, there exists $a_i \in A$ such that $\sqrt{(I : a_i)} = \sqrt{Q_i}$.

Q 5.
Prove that the zero ideal in the ring $C[0, 1]$ of real-valued continuous functions does not have a primary decomposition.
Hint : You may use 5(ii) and deduce that an element a_i as above may be non-zero at the most at one point.

Q 6.

Let A be Noetherian and let I be an ideal. Prove that $I(\bigcap_{n=1}^{\infty} I^n) = \bigcap_{n=1}^{\infty} I^n$.

Q 7.

(i) For any abelian group C, compute the abelian group $Tor_n(C, \mathbf{Z}_p)$ for all $n \geq 0$.

(ii) For ideals I, J show that the A-modules $Tor_1(A/I, A/J)$ and $Tor_2(A/I, A/J)$ are isomorphic to $(I \cap J)/IJ$ and $Ker(I \otimes J \to IJ)$ respectively.

Q 8.

For a Noetherian ring A, and a finitely generated A-module M, prove that $\bigcup\{P : P \in Ass(M)\}$ is the set of zero divisors of M.

Q 9.

Let $f, g : P_\bullet \to Q_\bullet$ be homotopic maps between two complexes of A-modules. Show that, for each $n \geq 0$, the induced homomorphisms f_n^*, g_n^* on the n-th homology are the same.